Nonlinear Circuits

Circuits with nonlinear components

Linear and Nonlinear Circuits, Chua, Desoer & Kuh 1987
Chapter 2 section 1.1, 1.2, 1.4, 2.1, 2.2, and 2.3

1.1 From Linear Resistor to Resistor

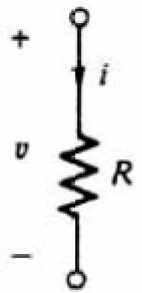


Figure 1.1 Symbol for a linear resistor with resistance R.

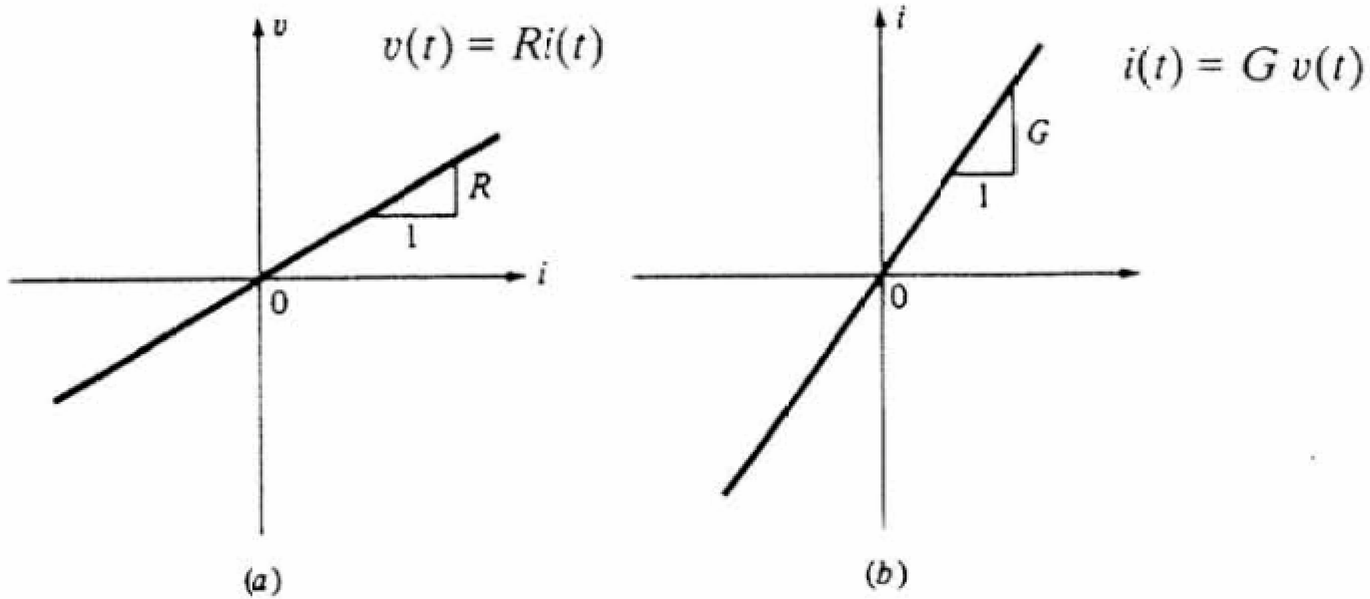


Figure 1.2 Linear resistor characteristic plotted (a) on the i-v plane and (b) on the v-i plane.

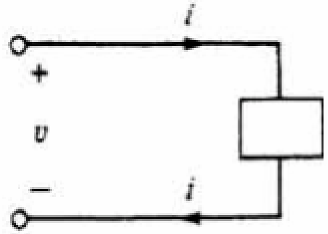


Figure 1.3 A two-terminal element with v and i in the associated reference directions.

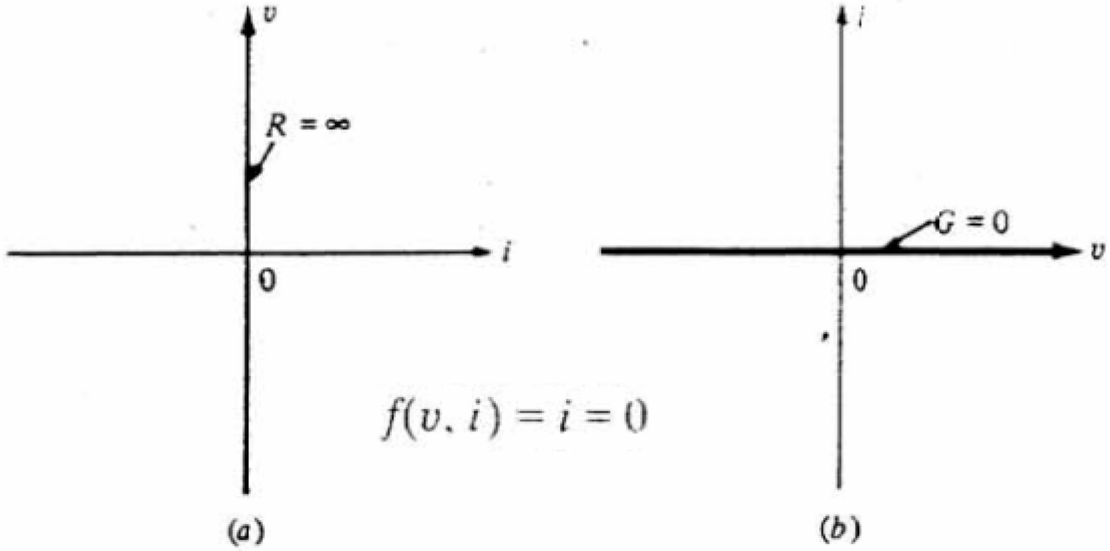
$$\mathcal{R}_R = \{(v, i): f(v, i) = 0\}$$

The linear resistor is a special case of a resistor in which

$$f(v, i) = v - Ri = 0 \qquad \text{or} \qquad f(v, i) = i - Gv = 0$$

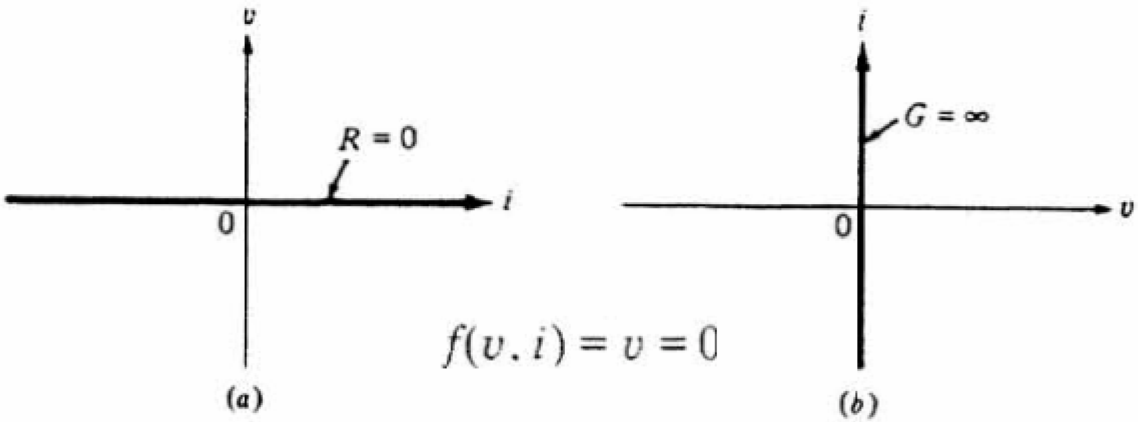
A resistor which is not linear is called *nonlinear*.

Open circuits and short circuits



$$f(v, i) = i = 0$$

Figure 1.4 Characteristic of an open circuit.



$$f(v, i) = v = 0$$

Figure 1.5 Characteristic of a short circuit.

Power, passive resistors, active resistors, and modeling

$$p(t) = v(t)i(t)$$

If the resistor is linear having resistance R

$$p(t) = Ri^2(t) = Gv^2(t)$$

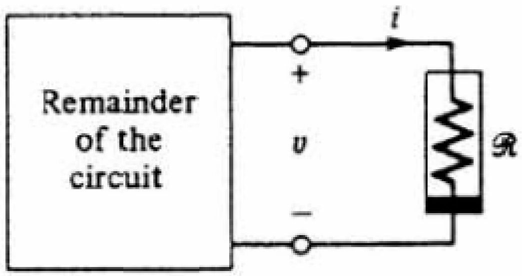


Figure 1.6 Illustrating power delivered to a nonlinear resistor from the remainder of the circuit.

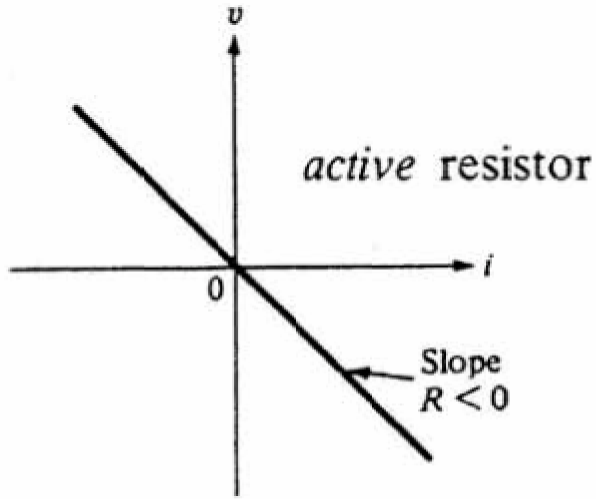


Figure 1.7 Characteristic of a linear active resistor with resistance $R < 0$.

1.2 The Nonlinear Resistor

pn-Junction diode

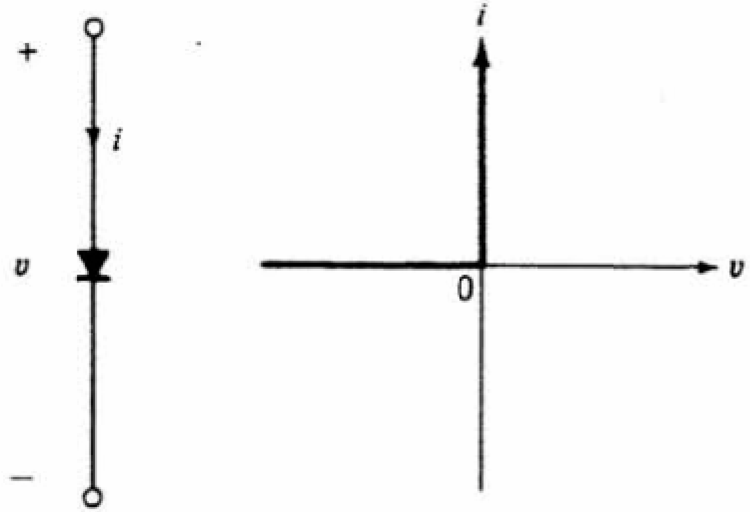


Figure 1.8 Symbol for an ideal diode and its characteristic.

$$\mathcal{R}_{\text{ID}} = \{(v, i)\colon vi = 0, i = 0 \text{ for } v < 0 \text{ and } v = 0 \text{ for } i > 0\}$$

$$i = I_s\left[\exp\left(\frac{v}{V_T}\right) - 1\right]$$

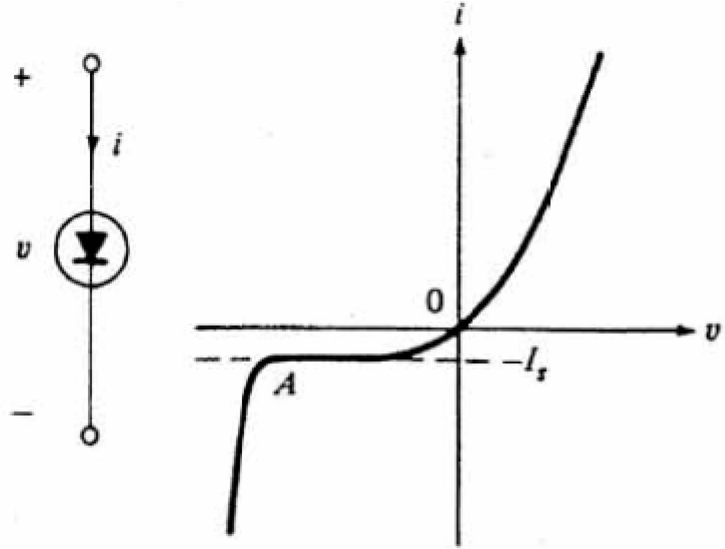


Figure 1.9 Symbol for a *pn*-junction diode and its characteristic.

Tunnel diode

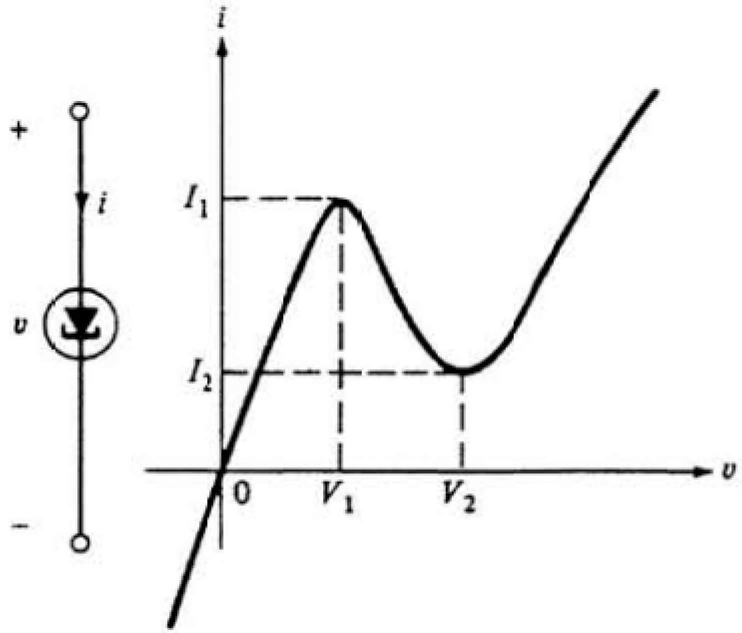
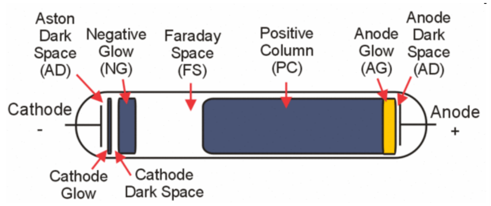


Figure 1.10 Symbol for a tunnel diode and its characteristic.

$$i = \hat{i}(v)$$

voltage-controlled nonlinear resistor

Glow tube



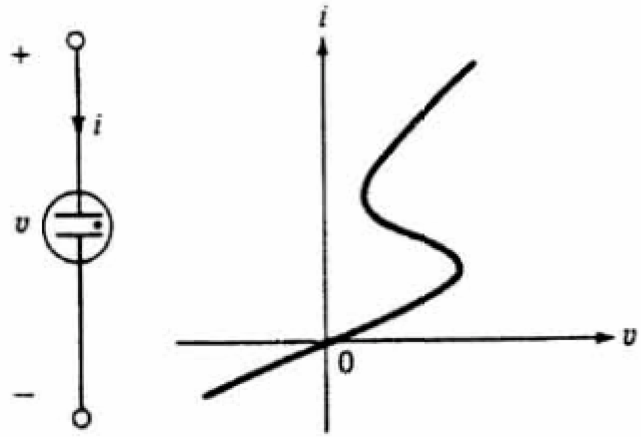


Figure 1.11 Symbol for a glow tube and its characteristic.

$$v = \hat{v}(i)$$

current-controlled resistor

Bilateral property

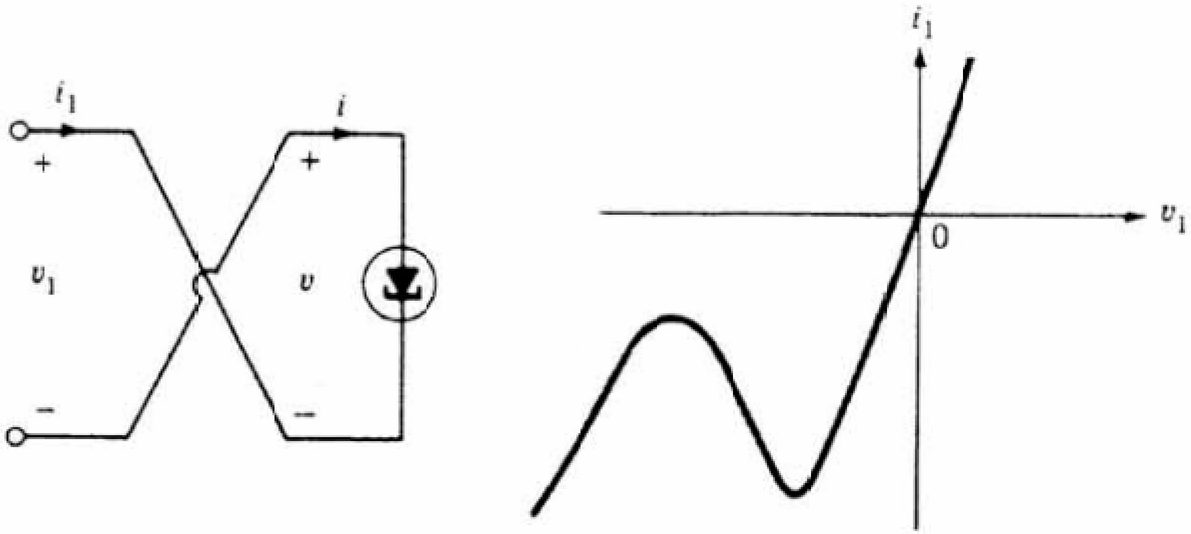


Figure 1.12 Characteristic of a tunnel diode with its terminals turned around.

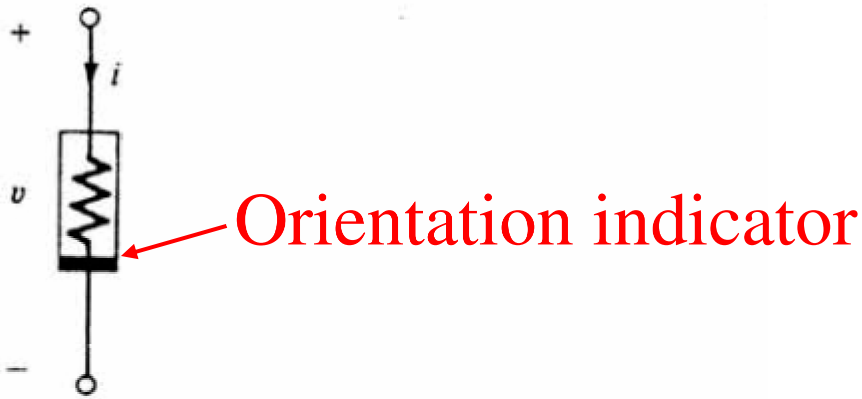


Figure 1.13 Symbol for a nonlinear resistor.

The characteristic of a *linear* resistor is always symmetric with respect to the origin. A circuit element with this kind of symmetry is called *bilateral*. A *bilateral resistor* satisfies the property $f(v, i) = f(-v, -i)$ for all (v, i) on its characteristic. A nonlinear resistor may be bilateral, e.g., have the characteristic shown in Fig. 1.14.

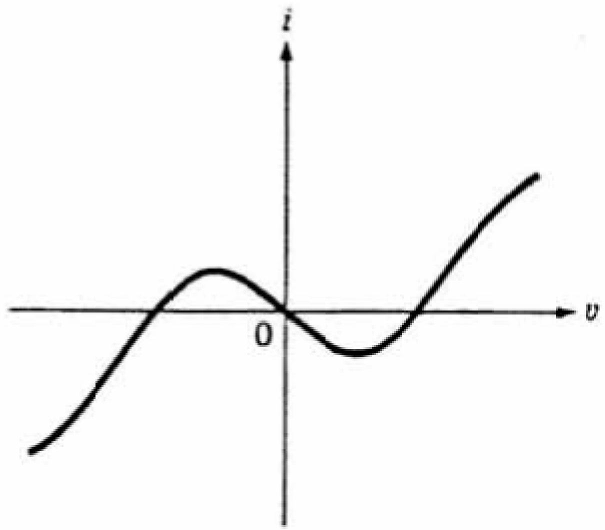
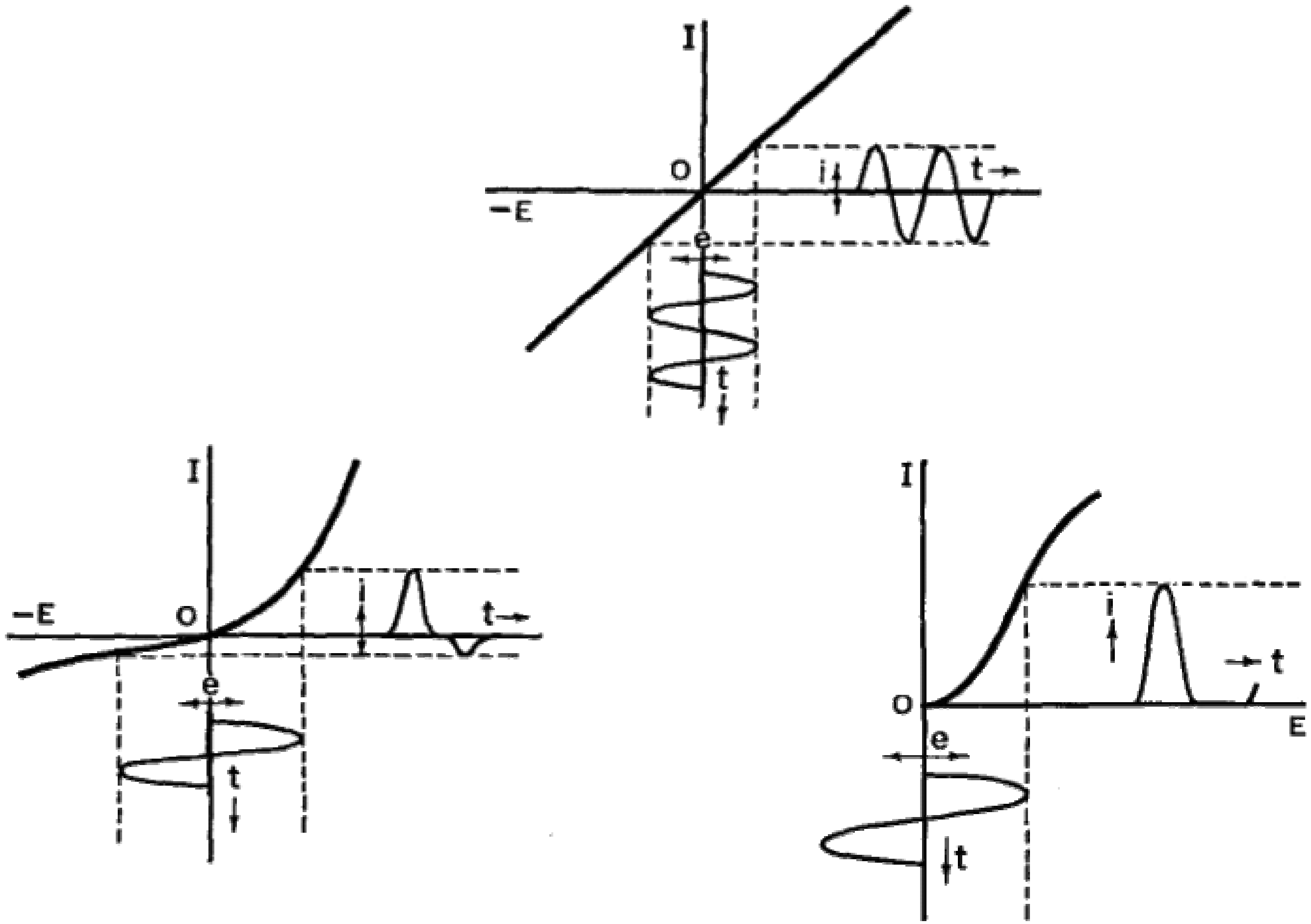
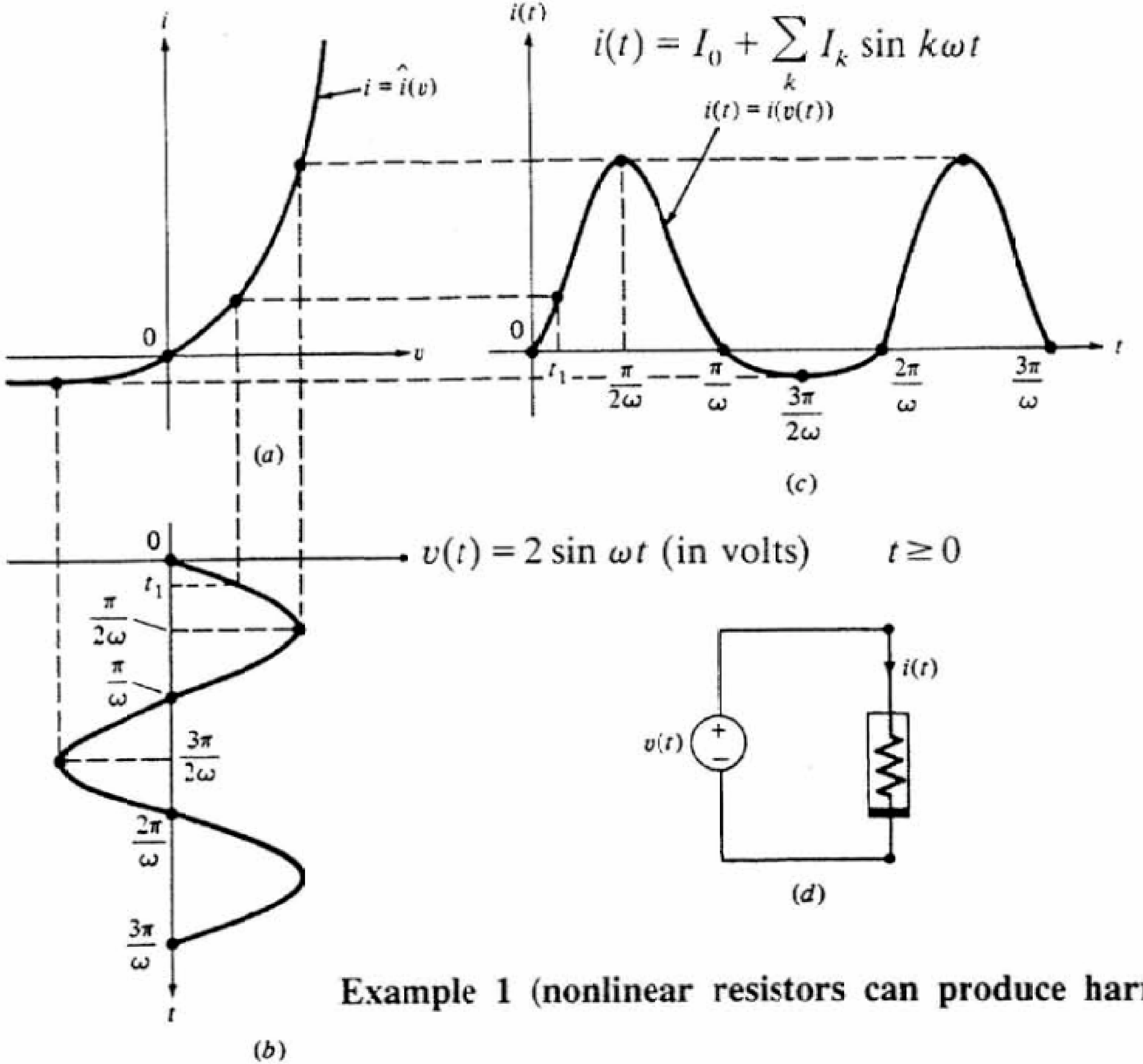


Figure 1.14 v-i Characteristic of a bilateral nonlinear resistor.

Simple circuits Circuits containing nonlinear resistors have properties totally different from those which have only linear resistors. The following examples illustrate some of the differences.





Example 1 (nonlinear resistors can produce harmonics)

Figure 1.15 An example illustrating a special clipping property of nonlinear resistors; the negative half of the waveform has been clipped.

Example 2 (piecewise-linear approximation)

$$i(t) = \begin{cases} 2G \sin \omega t & \text{for} \quad \dfrac{2n\pi}{\omega} \le t \le \dfrac{(2n+1)\pi}{\omega} \\ 0 & \text{for} \quad \dfrac{(2n+1)\pi}{\omega} \le t \le \dfrac{(2n+2)\pi}{\omega} \end{cases}$$

where n runs through nonnegative integers.

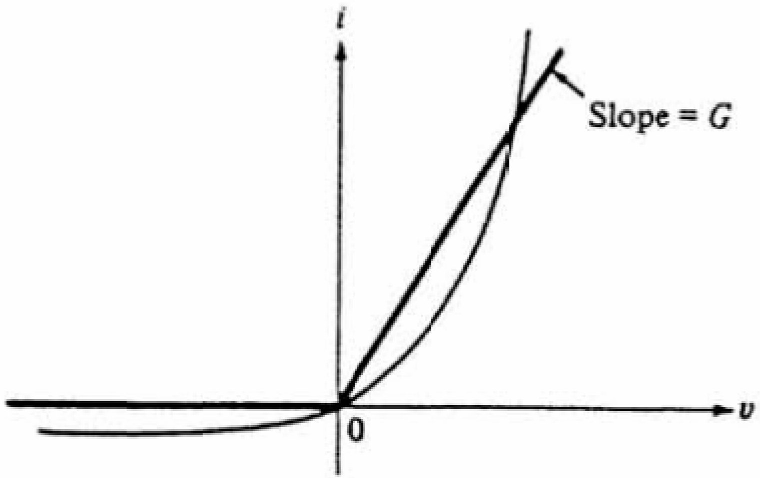


Figure 1.16 Piecewise-linear approximation of a nonlinear characteristic.

Example 3 (homogeneity and additivity)

the currents in a linear resistor whose characteristic is

$$i_\ell = \hat{i}_\ell(v) = 0.1v$$

and in a nonlinear resistor whose characteristic is

$$i_n = \hat{i}_n(v) = 0.1v + 0.02v^3$$

$v_1 = 1$ V, $v_2 = k$ V, and $v_3 = v_1 + v_2 = 1 + k$ V.

$$i_{\ell 2} = \hat{i}_\ell(kv_1) = k\hat{i}_\ell(v_1)$$
$$i_{\ell 3} = \hat{i}_\ell(v_1 + v_2) = \hat{i}_\ell(v_1) + \hat{i}_\ell(v_2)$$

$v_1 = 1$ V, $v_2 = k$ V, and $v_3 = v_1 + v_2$ V

$$i_{n2} = \hat{i}_n(kv_1) \neq k\hat{i}_n(v_1)$$
$$i_{n3} = \hat{i}_n(v_1 + v_2) \neq \hat{i}_n(v_1) + \hat{i}_n(v_2)$$

1.4 Time-Invariant and Time-Varying Resistors

$$v(t) = R(t)i(t) \qquad \text{or} \qquad i(t) = G(t)v(t)$$

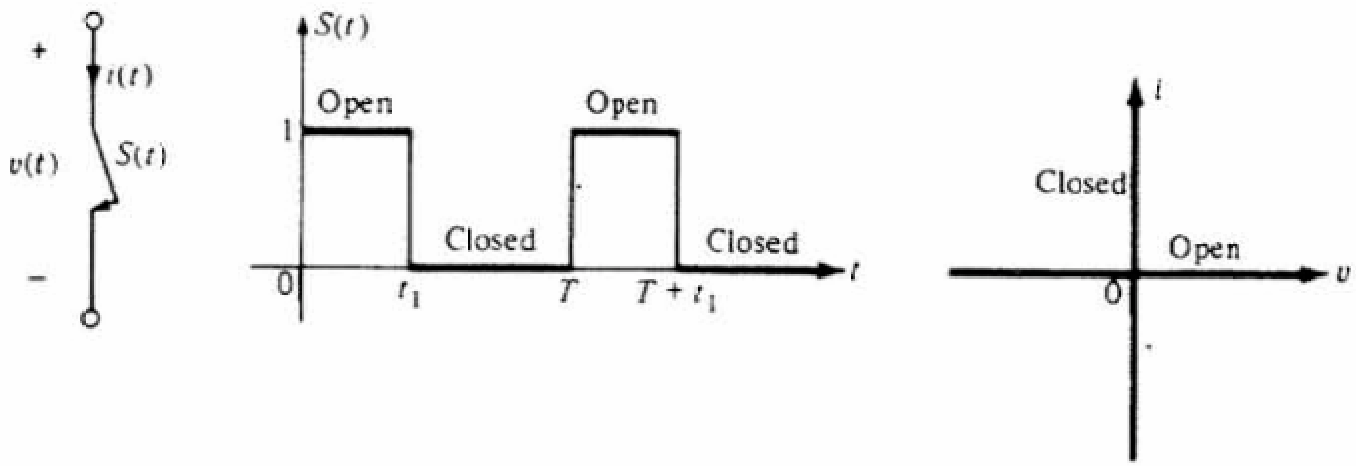


Figure 1.22 Symbol for a periodically operating switch and its characteristic. Here, $S(t)$ denotes the "state" (i.e., position) of the switch.

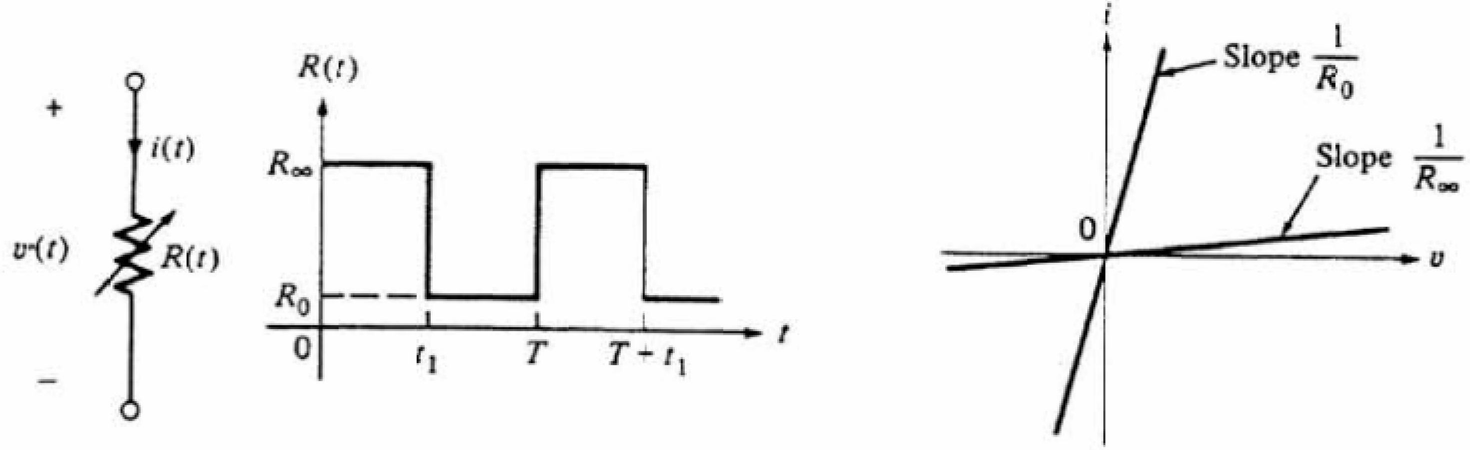


Figure 1.23 Symbol for a linear time-varying resistor and its characteristic.

Example Consider a linear time-varying resistor with v-i characteristic

$$i(t) = G(t)v(t) \qquad \text{and} \qquad G(t) = G_a + G_b \cos \omega t$$

where the constants G_a and G_b satisfy the condition $G_a > G_b$; consequently $G(t) > 0$ for all t. The angular frequency ω is also a constant. Let the voltage waveform be a sinusoid with angular frequency ω_1. Then

$$v(t) = V_m \cos \omega_1 t$$

$$\begin{aligned} i(t) &= G_a V_m \cos \omega_1 t + G_b V_m \cos \omega_1 t \cos \omega t \\ &= G_a V_m \cos \omega_1 t + \frac{G_b V_m}{2} \cos(\omega + \omega_1)t + \frac{G_b V_m}{2} \cos(\omega - \omega_1)t \end{aligned}$$

Generate 3 frequencies

2 SERIES AND PARALLEL CONNECTIONS

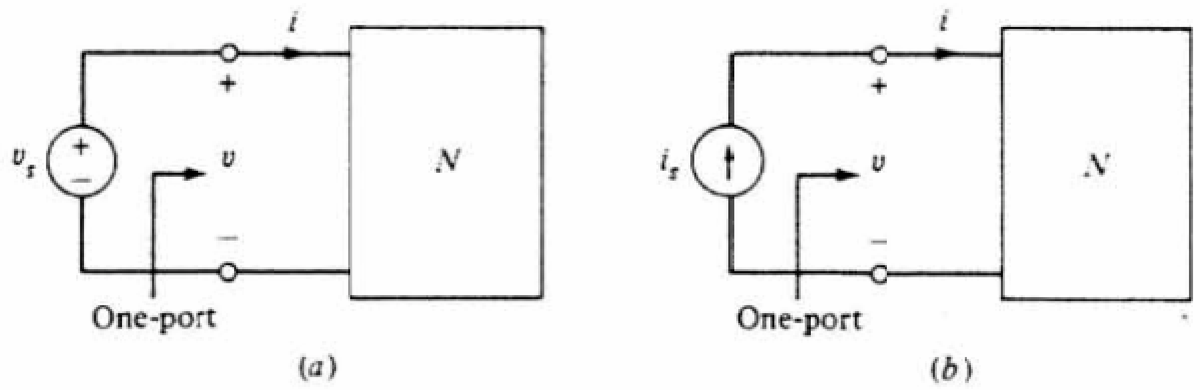
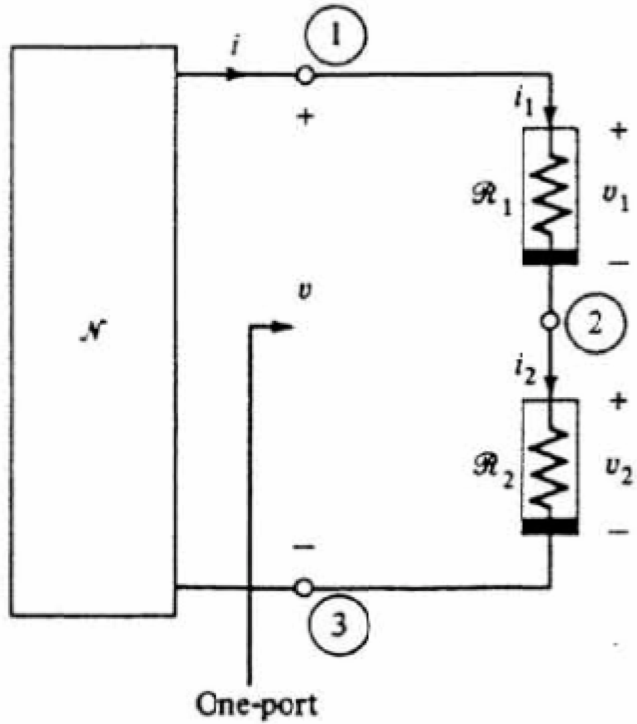


Figure 2.1 A one-port N driven (a) by an independent voltage source and (b) by an independent current source.

2.1 Series Connection of Resistors



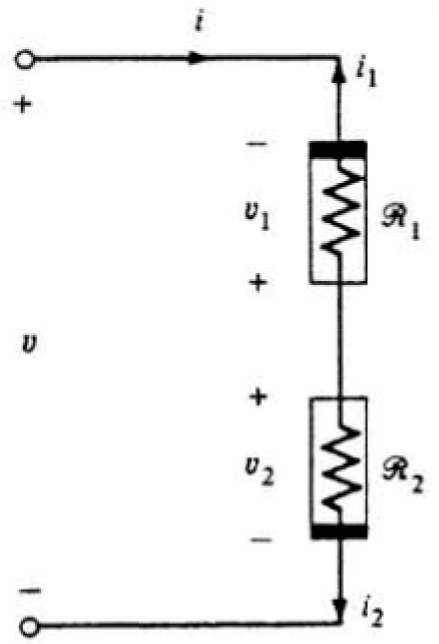
$$v_1 = \hat{v}_1(i_1) \qquad \text{and} \qquad v_2 = \hat{v}_2(i_2)$$

$$i = i_1 = i_2$$

$$v = v_1 + v_2$$

$$v = \hat{v}_1(i) + \hat{v}_2(i)$$

Figure 2.2 Two nonlinear resistors connected in series together with the rest of the circuit $\mathscr{N}$.



$$\hat{v}(i) = -\hat{v}_1(-i) + \hat{v}_2(i)$$

Figure 2.3 Series connection of $\mathcal{R}_1$ and $\mathcal{R}_2$ with the terminals of $\mathcal{R}_1$ turned around.

Example 1 (a battery model)

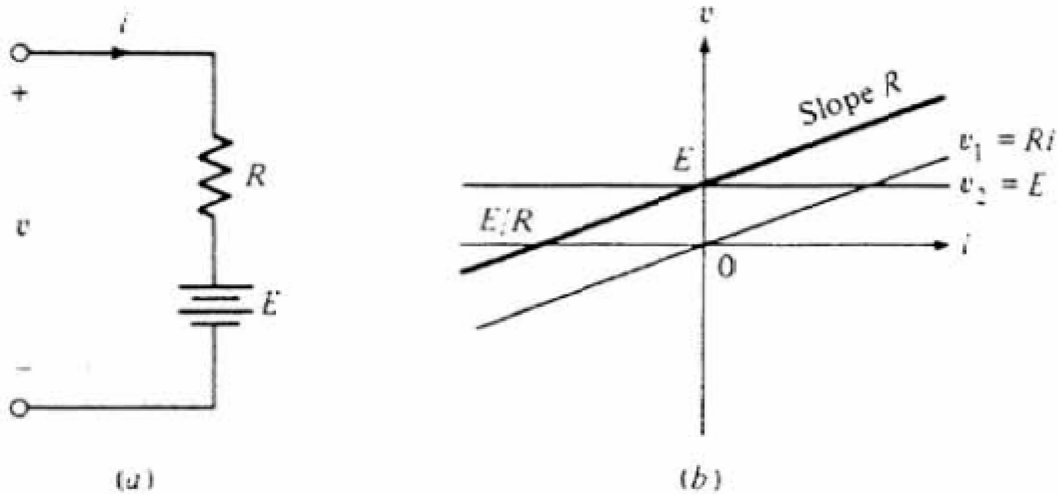


Figure 2.4 (*a*) Series connection of a linear resistor and a dc voltage source, and (*b*) its driving-point characteristic.

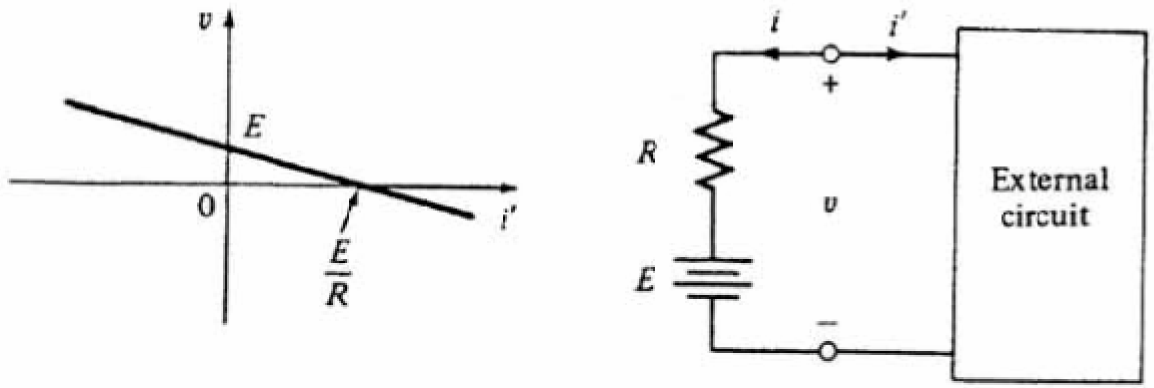


Figure 2.5 Characteristic of a real battery plotted on the i'-v plane and the external circuit connection.

Example 2 (ideal diode circuit)

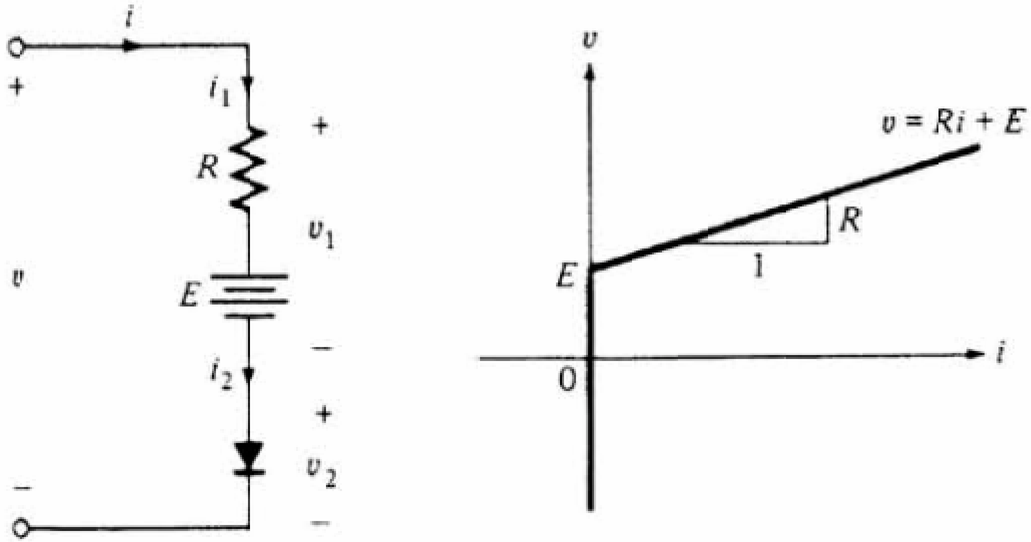


Figure 2.6 Series connection of a battery with an ideal diode and the driving-point characteristic of the resulting one-port.

$$v = v_1 = Ri + E \qquad \text{for } i > 0$$

$$v = E + v_2 \qquad \text{and} \qquad i = 0 \qquad \text{for } v_2 \leq 0$$

Example 5 (graphic method)

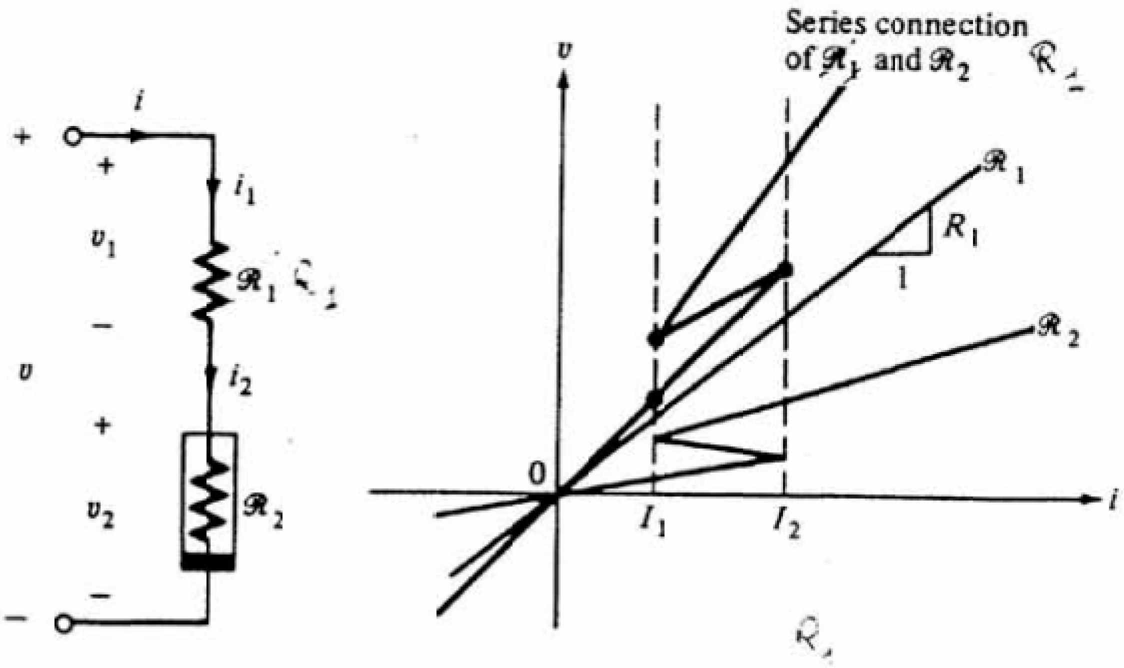


Figure 2.9 Series connection of a linear resistor $\mathscr{R}_1$ and a voltage-controlled resistor $\mathscr{R}_2$.

$$v_1 = R_1 i_1 \qquad \text{and} \qquad i_2 = \hat{i}_2(v_2)$$

Summary of series connection The key concepts used in obtaining the driving-point characteristic of a one-port formed by the series connection of two-terminal resistors are

1. KCL forces all branch currents to be equal to the port current.
2. KVL requires the port voltage v to be equal to the sum of the branch voltages of the resistors.
3. If each resistor is current-controlled, the resulting driving-point characteristic of the one-port is also a current-controlled resistor.

2.2 Parallel Connection of Resistors

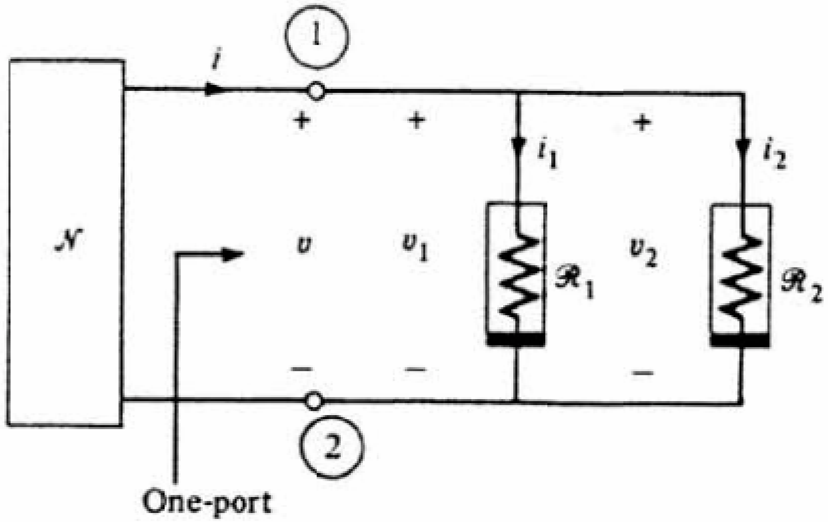


Figure 2.10 Two nonlinear resistors in parallel together with the rest of the circuit $\mathcal{N}$.

$$i_1 = \hat{i}_1(v_1) \qquad \text{and} \qquad i_2 = \hat{i}_2(v_2)$$

$$i = \hat{i}_1(v) + \hat{i}_2(v)$$

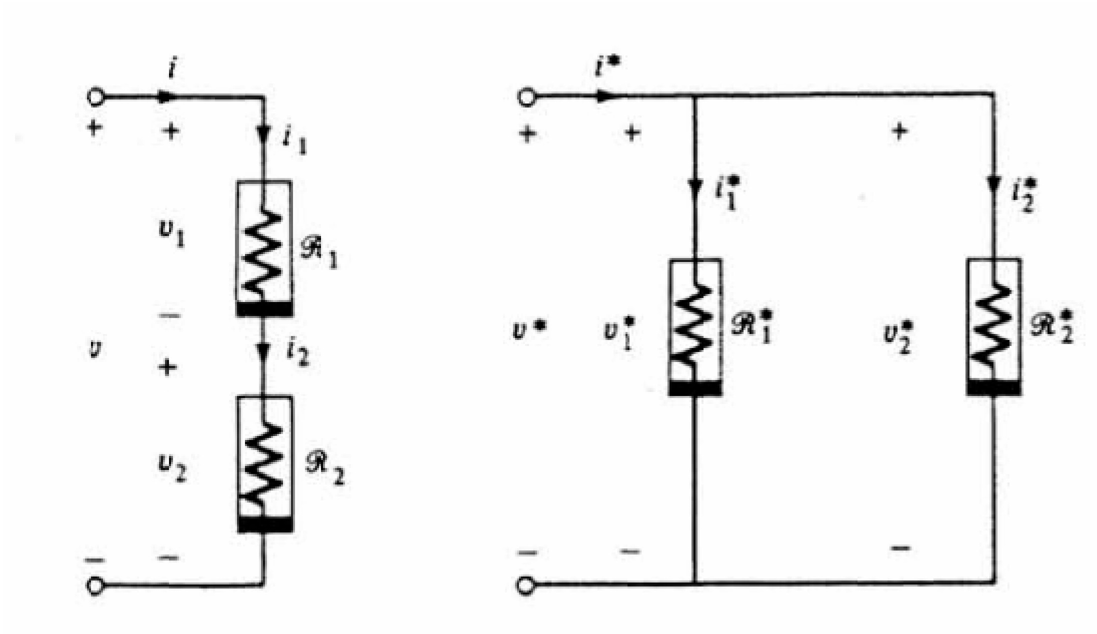


Figure 2.11 A circuit $\mathcal{N}$ and its dual $\mathcal{N}^*$.

Table 2.1 Dual terms

S	S^*
Branch voltage	Branch current
Current-controlled resistor	Voltage-controlled resistor
Resistance	Conductance
Open circuit	Short circuit
Independent voltage source	Independent current source
Series connection	Parallel connection
KVL	KCL
Port voltage	Port current

Example 1 (dual one-port and equivalent one-port)

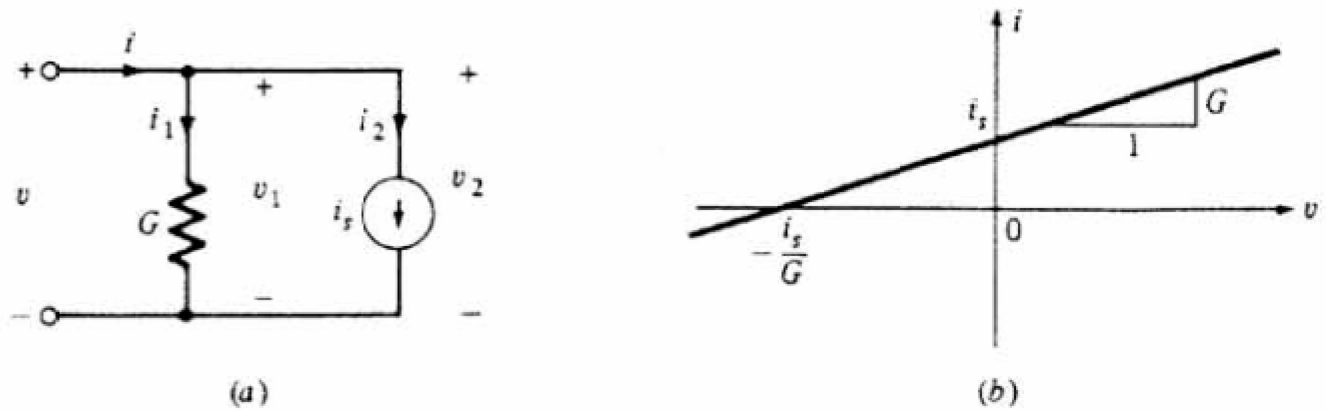


Figure 2.12 (*a*) Parallel connection of a linear resistor and an independent current source. (*b*) The driving-point characteristic of the resulting one-port.

$$i_1 = Gv_1 \qquad \text{and} \qquad i_2 = i_s$$

$$i = Gv + i_s$$

$$R'i = v + R'i_s$$

$$v = R'i - v'_s$$

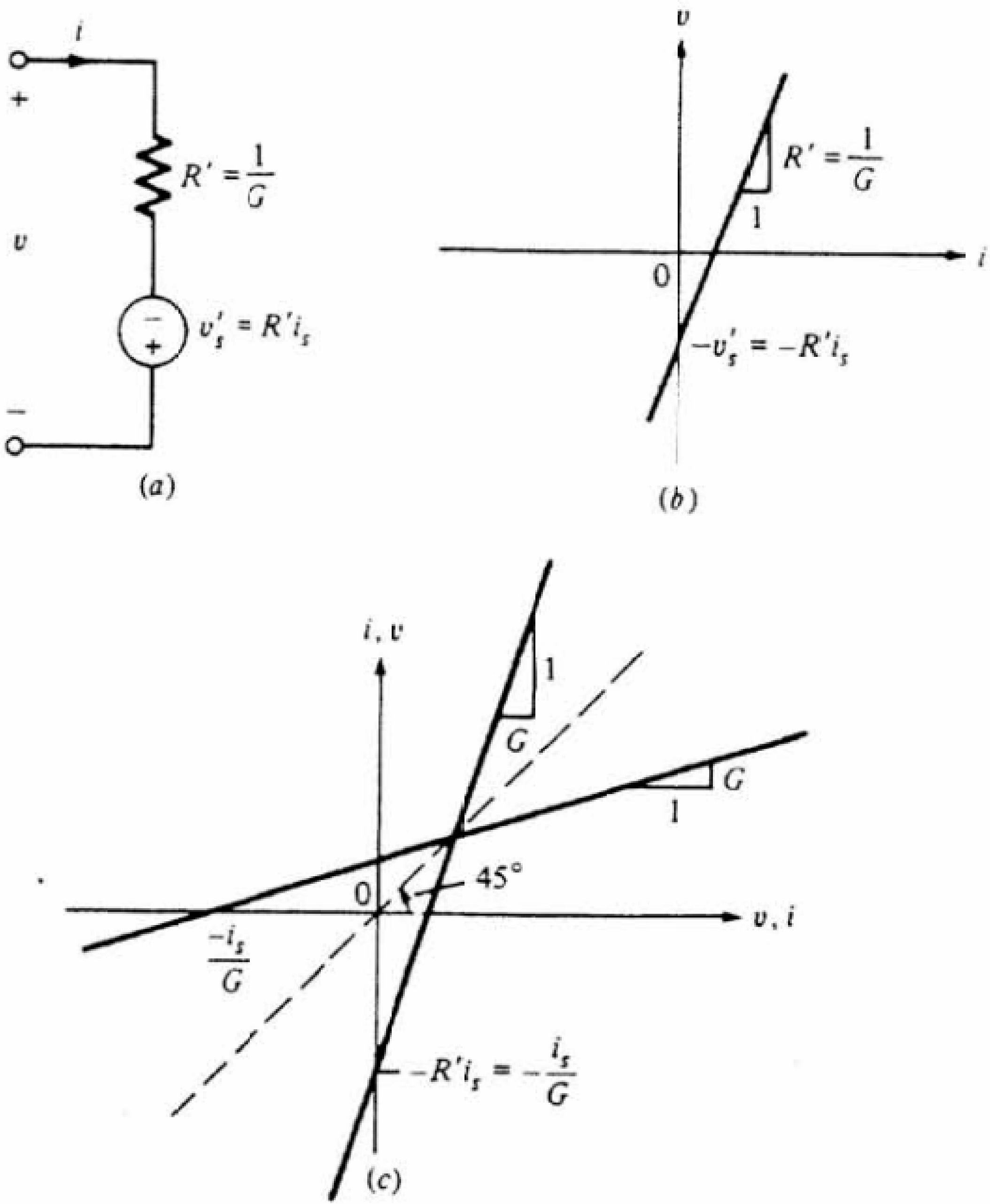


Figure 2.13 The one-port in (*a*) is equivalent to that of Fig. 2.12*a* since they have the same driving-point characteristics.

Example 2 (more on the ideal diode)

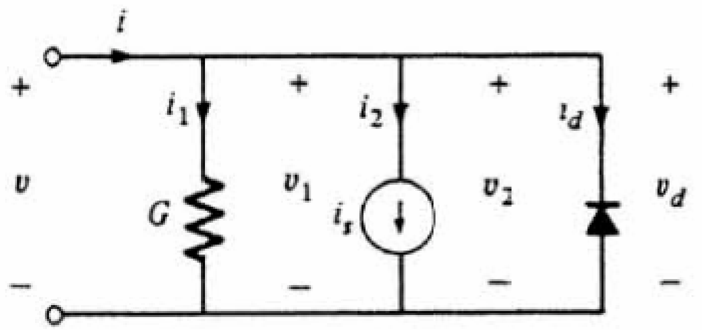


Figure 2.14 Parallel connection of a linear resistor, a current source, and an ideal diode.

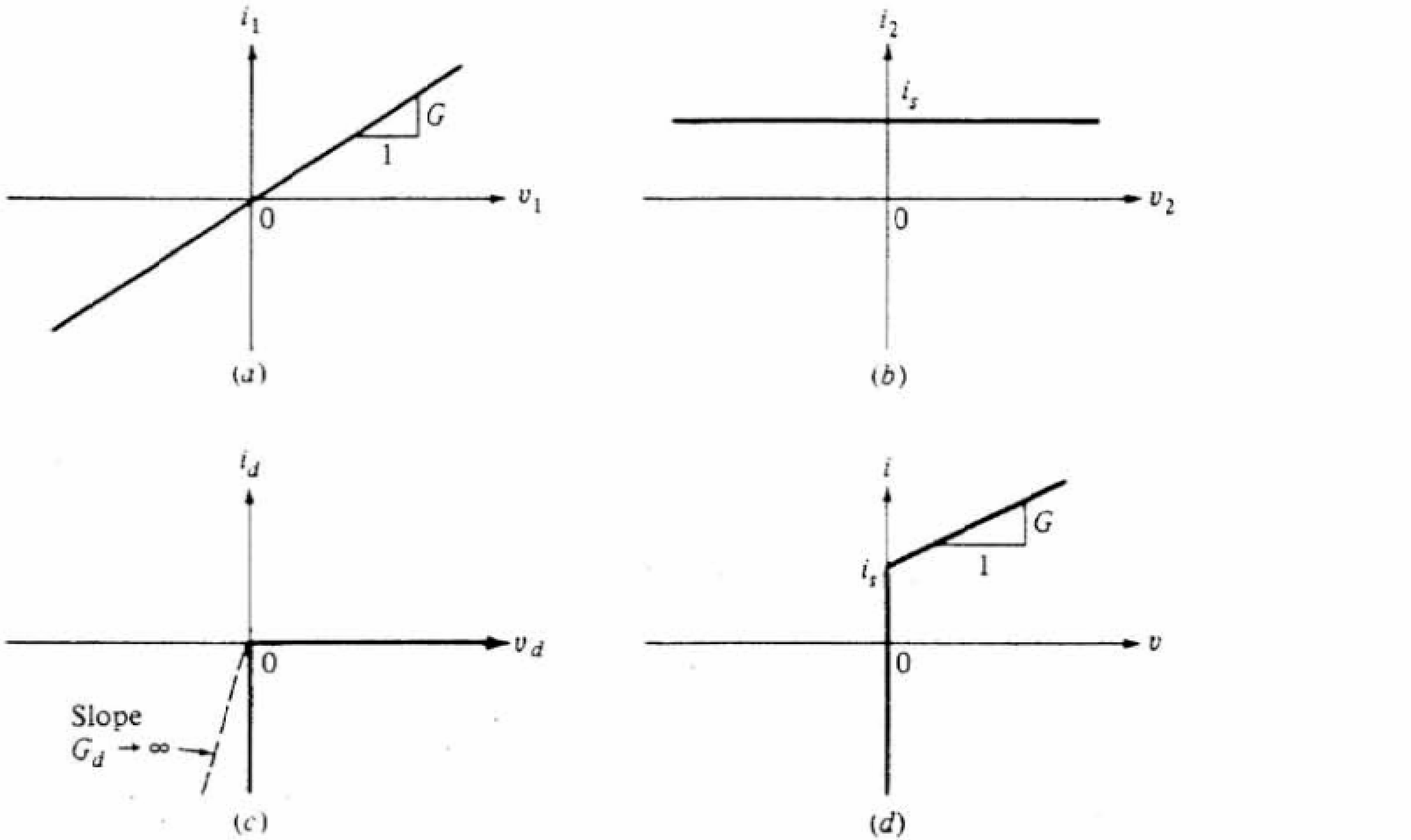


Figure 2.15 Branch v-i characteristics of (a) a linear resistor, (b) a current source, and (c) an ideal diode. (d) The resulting one-port characteristic.

Summary of parallel connection The key concepts used in obtaining the driving-point characteristic of a one-port formed by the parallel connection of two-terminal resistors are

1. KVL forces all branch voltages to be equal.
2. KCL requires the port current i to be equal to the sum of the branch currents of the resistors.
3. If each resistor is voltage-controlled, the resulting driving-point characteristic of the one-port is also a voltage-controlled resistor.

2.3 Series-Parallel Connection of Resistors

Example 1 (series-parallel connection of linear resistors)

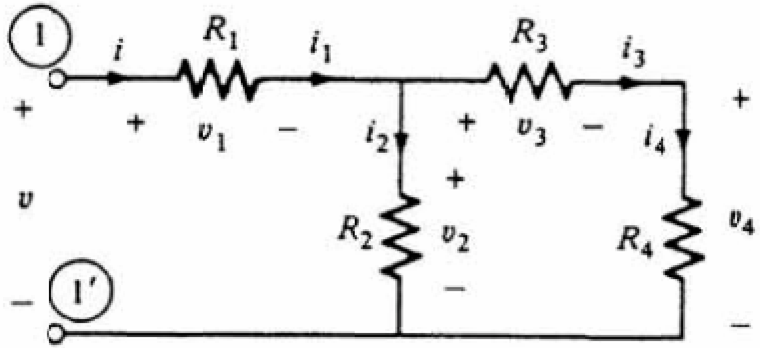


Figure 2.17 A ladder circuit with linear resistors.

$$v_2 = v_3 + v_4 = (R_3 + R_4)i_3$$

$$i_1 = i_2 + i_3 = G_2 v_2 + \frac{1}{R_3 + R_4} v_2 = \left(G_2 + \frac{1}{R_3 + R_4}\right) v_2$$

$$v = v_1 + v_2 = R_1 i_1 + \frac{1}{G_2 + 1/(R_3 + R_4)} i_1$$

$$v = Ri$$

where

$$R \triangleq R_1 + \frac{1}{G_2 + 1/(R_3 + R_4)}$$

Example 2 (series-parallel connection of nonlinear resistors)

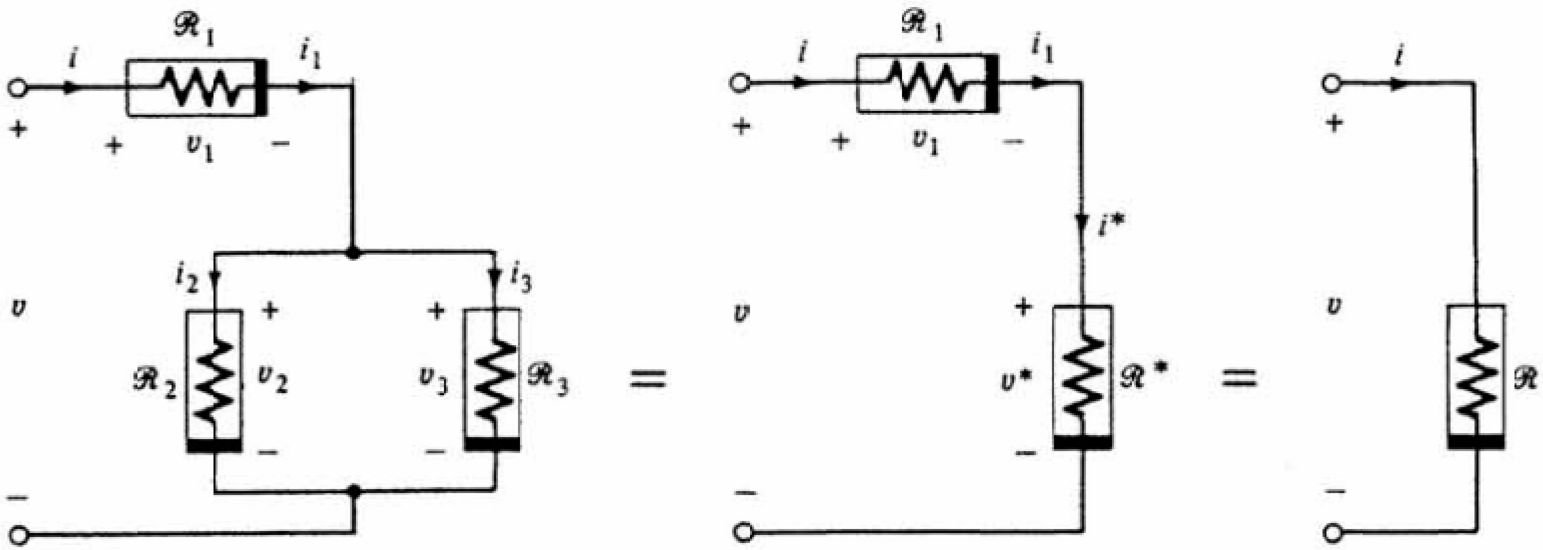


Figure 2.19 Reduction of a ladder circuit with nonlinear resistors.

$\mathcal{R}_2$ and $\mathcal{R}_3$ are voltage-controlled

$$i_2 = \hat{i}_2(v_2) \qquad \text{and} \qquad i_3 = \hat{i}_3(v_3)$$

$$i^* = g(v^*)$$

$$g(v^*) = \hat{i}_2(v^*) + \hat{i}_3(v^*) \qquad \text{for all } v^*$$

$\mathscr{R}_1$ is current-controlled

$$v_1 = \hat{v}_1(i_1)$$

$$v^* = g^{-1}(i^*) \qquad \text{for all } i^*$$

$$i = i_1 = i^* \text{ and } v = v_1 + v^*$$

$$v = \hat{v}(i)$$

where
$$\hat{v}(i) = \hat{v}_1(i) + g^{-1}(i) \qquad \text{for all } i$$

In this problem, one key step is the determination of the inverse function $g^{-1}(\cdot)$. The question is therefore whether the inverse exists. If it does not, the characteristic of $\mathcal{R}$ cannot be written as in Eq. (2.24) because it is not current-controlled. One simple criterion which guarantees the existence of the inverse is that the v-i characteristic is *strictly monotonically increasing*, i.e., the slope, $g'(v^*)$ is positive for all v^*

REMARK The characteristic of the one-port shown in Fig. 2.19 can always be represented parametrically. Indeed, we have

$$i = i_1 = i^* \qquad \text{and} \qquad v = v_1 + v^*$$

Hence, using v^* as a parameter, we obtain

$$i = g(v^*)$$
$$v = \hat{v}_1[g(v^*)] + v^*$$

1 TWO-TERMINAL CAPACITORS AND INDUCTORS

1.1 *q*-*v* and *ϕ*-*i* characteristics

$$f_C(q, v) = 0$$

$$f_L(\phi, i) = 0$$

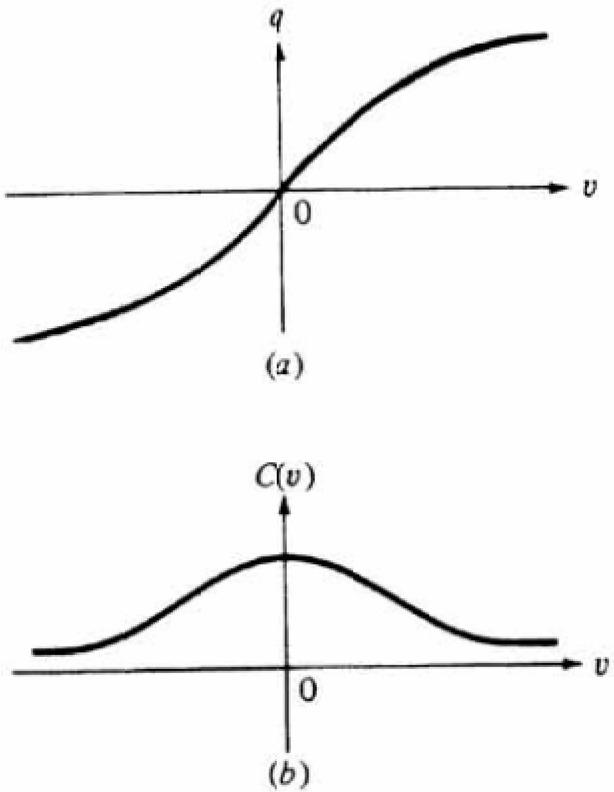


Figure 1.3 Nonlinear *q*-*v* characteristic.

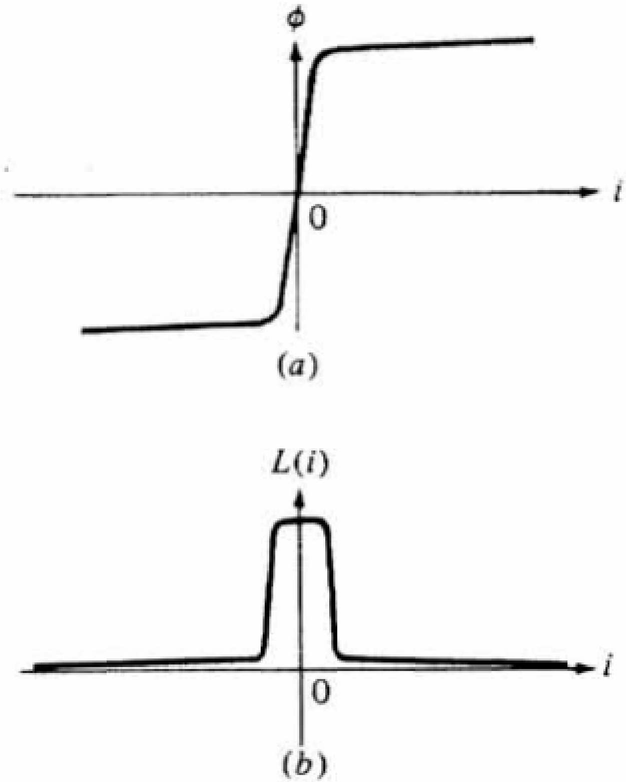


Figure 1.4 Nonlinear *ϕ*-*i* characteristic.

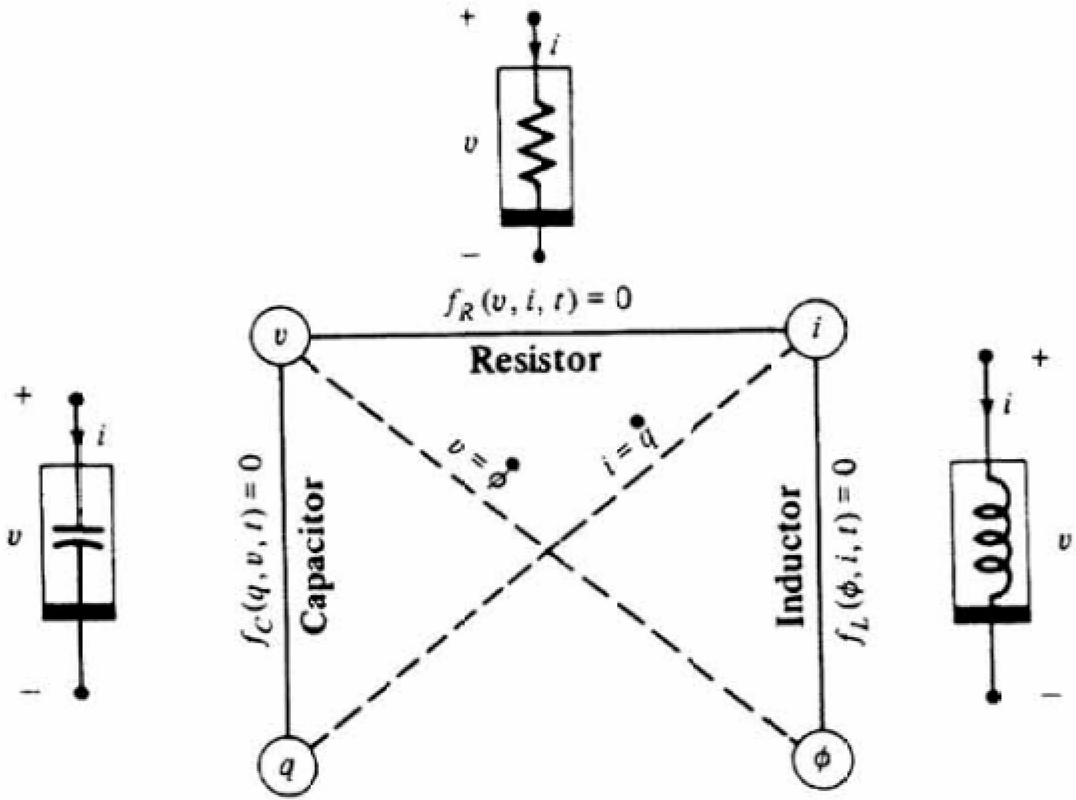


Figure 1.9 Basic circuit element diagram.

[11] A fourth nonlinear two-terminal element called the *memristor* is defined by the remaining relationship between q and ϕ. This circuit element is described in L. O. Chua, "Memristor—The Missing Circuit Element," *IEEE Trans. on Circuit Theory*, vol. 18, pp. 507–519, September 1971.